View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

Maximal Lyapunov Scaling Factors and Their Applications in the Study of Lyapunov Diagonal Semistability of Block Triangular Matrices

Dafna Shasha and Daniel Hershkowitz*

Mathematics Department
Technion — Israel Institute of Technology
Haifa 32000, Israel

Submitted by Hans Schneider

ABSTRACT

For a Lyapunov diagonally semistable matrix A, we introduce the concept of maximal Lyapunov scaling factor, that is, a Lyapunov scaling factor D of A such that the range of $AD+DA^T$ is of maximal dimension. We investigate properties of maximal Lyapunov scaling factors, and apply them in proving a characterization for block triangular Lyapunov diagonally semistable matrices in terms of Lyapunov diagonal semistability of the diagonal blocks. The latter result is used to prove that a block triangular matrix is Lyapunov diagonally semistable if and only if every 2×2 block principal submatrix of it is Lyapunov diagonally semistable.

1. INTRODUCTION

A square complex matrix is (positive) *stable* if its spectrum lies in the open right half plane. Lyapunov characterized the stable matrices by proving that a matrix A is stable if and only if there exists a positive definite (Hermitian) matrix H such $AH+HA^*$ is positive definite. That is why stable matrices are often called *Lyapunov stable*. The specific case where the matrix H above can be chosen to be diagonal plays an important role in

*The research of this author was supported in part by the Technion VPR grant No. 100-0678.

© Elsevier Science Publishing Co., Inc., 1988
52 Vanderbilt Ave., New York, NY 10017 0024-3795/88/$3.50

various disciplines and has motivated a rich mathematical literature. We call a square real matrix A *Lyapunov diagonally stable* [*Lyapunov diagonally semistable*] if there exists a positive definite diagonal matrix D, called a *Lyapunov scaling factor of* A, such that the matrix $AD + DA^T$ is positive definite [positive semidefinite]. The problem of characterizing Lyapunov diagonal semistability is in general hard.

In this paper we discuss Lyapunov diagonally semistable matrices. For a given such matrix A we define a maximal Lyapunov scaling factor of A to be a Lyapunov scaling factor D of A such that the range of $AD + DA^T$ is of maximal dimension. The latter range is called the Lyapunov range of A. In Section 3 we prove several interesting properties of maximal Lyapunov scaling factors. For example, we show that if D_0 is a maximal Lyapunov scaling factor of A and if D is any other Lyapunov scaling factor of A, then the range of $AD + DA^T$ is contained in the range of $AD_0 + D_0A^T$. Also, the matrix $D_0 + D$ is a maximal Lyapunov scaling factor of A. Finally, D is a maximal Lyapunov scaling factor of A if and only if D^{-1} is a maximal Lyapunov scaling factor of A^T.

In Section 4 we discuss block triangular matrices with square diagonal blocks. The observations of Section 3 are used here to prove that such a matrix is Lyapunov diagonally semistable if and only if the diagonal blocks are Lyapunov diagonally semistable and the ranges of the off-diagonal blocks are contained in the Lyapunov ranges of the diagonal blocks. The latter condition is related to but somewhat stronger than the weak principal submatrix rank property, proven in [3] to be shared by all Lyapunov diagonally semistable matrices. This result is then applied to prove that a $p \times p$ block triangular matrix A is Lyapunov diagonally semistable if and only if every 2×2 principal block submatrix of A is Lyapunov diagonally semistable. This theorem seems to have the potential of being very useful in the study of reducible Lyapunov diagonally semistable matrices, since it reduces the problem of characterizing Lyapunov diagonally semistable matrices with p diagonal blocks in the Frobenius normal form to the case $p = 2$.

Some of the assertions proved in Section 4 are applied in Section 5 in a further investigation of maximal Lyapunov scaling factors and Lyapunov ranges. In particular we show that D is a maximal Lyapunov scaling factor of A if and only if D is a direct sum of maximal Lyapunov scaling factors of the diagonal blocks, and a certain additional condition holds. Also, we show that the dimension of the Lyapunov range of A equals the sum of the dimensions of the Lyapunov ranges of the diagonal blocks. We conclude the paper with some observations on the inclusion relations between the range of A and the Lyapunov range of A.

2. NOTATION AND DEFINITIONS

This section contains most of the notation and definitions to be used in the paper.

NOTATION 2.1. For a positive integer n we denote by:

$\langle n \rangle$ the set $\{1,2,\dots,n\}$,
R^n the set of all real n-dimensional (column) vectors.

NOTATION 2.2. For a set α we denote by $|\alpha|$ the cardinality of α.

DEFINITION 2.3. For a real number a we define the function $\operatorname{sign} a$ by

$$\operatorname{sign} a = \begin{cases} 1, & a > 0, \\ -1, & a < 0, \\ 0, & a = 0. \end{cases}$$

NOTATION 2.4. Let $x \in R^n$. We denote by $\operatorname{supp}(x)$ the support of x, that is, the set $\{i \in \langle n \rangle : x_i \neq 0\}$.

NOTATION 2.5. Let A be an $n \times n$ matrix, and let α and β be nonempty subsets of $\langle n \rangle$. We denote by:

$N(A)$ the nullspace (kernel) of A,
$\operatorname{Range}(A)$ the range of A,
$\operatorname{Rank}(A)$ the rank of A,
$\operatorname{col}(A)$ the set of columns of A (as a subset of R^n).
$A[\alpha|\beta]$ the submatrix of A whose rows are indexed by α and whose columns are indexed by β in their natural order.

We further denote

$$A[\alpha] = A[\alpha|\alpha],$$

$$A(\alpha) = A[\langle n \rangle \setminus \alpha].$$

Convention 2.6. Let A be an $n \times n$ matrix. The determinant of $A[\varnothing]$ is defined to have the value 1.

DEFINITION 2.7. A diagonal matrix is said to be a *positive diagonal matrix* if its diagonal entries are all positive.

DEFINITION 2.8 (see [3]). An $n \times n$ matrix A is said to have the *principal submatrix rank property* if it satisfies

$$\operatorname{Rank}(A[\alpha|\langle n\rangle]) = \operatorname{Rank}(A[\langle n\rangle|\alpha]) = \operatorname{Rank}(A[\alpha])$$

for all nonempty subsets α of $\langle n\rangle$.

DEFINITION 2.9 (see [3]). An $n \times n$ matrix A is said to have the *weak principal submatrix rank property* if it satisfies

$$A[j|\alpha] = 0 \quad \Rightarrow \quad A[\alpha|j] \in \operatorname{Range}(A[\alpha]),$$

$$A[\alpha|j] = 0 \quad \Rightarrow \quad A[j|\alpha]^T \in \operatorname{Range}\left(A[\alpha]^T\right)$$

for all $j \in \langle n\rangle \setminus \alpha$, for all nonempty subsets α of $\langle n\rangle$.

DEFINITION 2.10.

(i) A square matrix A is said to be in *Frobenius normal form* if A may be written in a block triangular form, where the diagonal blocks are irreducible square matrices.

(ii) Let A and B be square matrices. The matrix B is said to be a *Frobenius normal form of* A if B is in Frobenius normal form and if there exists a permutation matrix P such that $B = P^TAP$ (i.e., A and B are permutationally similar).

REMARK 2.11. Observe that by Definition 2.10 the Frobenius normal form of a square matrix A is unique up to permutation similarity, and so Frobenius normal forms of A have the same diagonal blocks up to permutation similarity.

3. MAXIMAL LYAPUNOV SCALING FACTORS

We start with a summary of standard properties of vector spaces and orthogonal complements that are going to be used in the sequel. The list is given here for reference.

PROPOSITION (3.1).

(i) *Let V and W be subspaces of a given vector space. Then*

$$(V \cap W)^{\perp} = V^{\perp} + W^{\perp}.$$

(ii) *Let V and W be subspaces of a given vector space. Then*

$$V \subseteq W \quad \Rightarrow \quad W^{\perp} \subseteq V^{\perp}.$$

(iii) *For every real square matrix C we have*

$$\mathrm{Range}(C^T) = N(C)^{\perp}.$$

The following lemma is essentially well known. A proof is provided for the sake of completeness.

LEMMA 3.2. *Let A and B be positive semidefinite real matrices. Then*

(i) $N(A+B) = N(A) \cap N(B)$,
(ii) $\mathrm{Range}(A) + \mathrm{Range}(B) = \mathrm{Range}(A+B)$.

Proof. (i): Since A, B and $A+B$ are positive semidefinite, it follows that

$$(3.3) \qquad x \in N(A) \quad \Leftrightarrow \quad x^T A x = 0,$$

$$(3.4) \qquad x \in N(B) \quad \Leftrightarrow \quad x^T B x = 0,$$

$$(3.5) \qquad x \in N(A+B) \quad \Leftrightarrow \quad x^T (A+B) x = 0.$$

Observe that

$$(3.6) \qquad x^T (A+B) x = x^T A x + x^T B x.$$

Since A and B are positive semidefinite, it follows that $x^T A x,\ x^T B x \geq 0$, and hence, by (3.6),

$$(3.7) \qquad x^T (A+B) x = 0 \quad \Leftrightarrow \quad x^T A x = x^T B x = 0.$$

Statement (i) now follows from (3.3), (3.4), (3.5), and (3.7).

(ii): By statements (i) and (iii) of Proposition (3.1), our statement (i) implies (ii). ■

An interesting corollary of Lemma 3.2 is the following.

COROLLARY 3.8. *Let A be a Lyapunov diagonally semistable matrix, and let D_1 and D_2 be Lyapunov scaling factors of A. If*

$$N(AD_1 + D_1A^T) \cap N(AD_2 + D_2A^T) = \{0\},$$

then A is Lyapunov diagonally stable. In particular, the matrix $A(D_1 + D_2) + (D_1 + D_2)A^T$ is positive definite.

Proof. Clearly, $D_1 + D_2$ is a Lyapunov scaling factor of A. Our claim follows by applying part (i) of Lemma 3.2 to the matrices $AD_1 + D_1A^T$ and $AD_2 + D_2A^T$. ■

NOTATION (3.9). Let A be a Lyapunov diagonally semistable matrix, and let D be a Lyapunov scaling factor of A. We denote the vector space $\mathrm{Range}(AD + DA^T)$ by $V(D, A)$, and the vector space $N(AD + DA^T)$ by $N(D, A)$.

THEOREM 3.10. *Let A be a Lyapunov diagonally semistable matrix, and let D_0 be a Lyapunov scaling factor of A such that $V(D_0, A)$ is of maximal dimension. Then for every Lyapunov scaling factor D of A we have $V(D, A) \subseteq V(D_0, A)$.*

Proof. Let D_0 be a Lyapunov scaling factor of A such that the dimension of $V(D_0, A)$ is maximal, and let D be any other Lyapunov scaling factor of A. Assume that $V(D, A) \not\subseteq V(D_0, A)$. Then the dimension of $W = V(D, A) + V(D_0, A)$ is greater than the dimension of $V(D_0, A)$. However, by part (ii) of Lemma 3.2, $W = V(D + D_0, A)$. Since obviously $D + D_0$ is a Lyapunov scaling factor of A, we now have a contradiction to the assumption that the dimension of $V(D_0, A)$ is maximal. Therefore, our assumption that $V(D, A) \not\subseteq V(D_0, A)$ is false. ■

Motivated by Theorem 3.10, we now define

DEFINITION 3.11. Let A be a Lyapunov diagonally semistable matrix. A Lyapunov scaling factor D_0 of A is said to be *maximal* if for every Lyapunov scaling factor D of A we have $V(D, A) \subseteq V(D_0, A)$. The vector space $V(D_0, A)$ is called the *Lyapunov range* of A, and is denoted by V_A.

COROLLARY 3.12. *Let A be a Lyapunov diagonally semistable matrix, and let D_0 be a maximal Lyapunov scaling factor of A. Then for every Lyapunov scaling factor D of A we have $N(D_0, A) \subseteq N(D, A)$.*

Proof. Our assertion follows from Theorem 3.10 by statements (ii) and (iii) of Proposition 3.1. ∎

EXAMPLE 3.13. Let A be the matrix

$$\begin{bmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

The matrix A is Lyapunov diagonally semistable, since $A + A^T$ is positive semidefinite. As is well known, Lyapunov diagonally stable matrices have positive principal minors. Thus, since A is singular, A is not Lyapunov diagonally stable, and hence the dimension of $V(D, A)$, whenever D is a Lyapunov scaling factor of A, cannot acceed 2. Indeed, the matrix $D = \operatorname{diag}(2, 2, 1)$ is a Lyapunov scaling factor of A, and since $V(D, A) = \{(x, x, y)^T : x, y \in R\}$ is 2-dimensional, it follows that D is a maximal Lyapunov scaling factor of A and that $V(D, A)$ is the Lyapunov range of A. Another Lyapunov scaling factor of A is the identity matrix. Observe that $V(I, A) = \{(x, x, x)^T : x \in R\}$ is a 1-dimensional subspace of $V(D, A)$. Also, it is easy to verify that $N(D, A)$ is contained in $N(I, A)$.

We conclude the section with a few observations on maximal Lyapunov scaling factors.

PROPOSITION 3.14. *Let A be a Lyapunov diagonally semistable matrix. Let D_0 be a maximal Lyapunov scaling factor of A, and let D be any Lyapunov scaling factor of A. Let $B_0 = AD_0 + D_0A^T$ and $B = AD + DA^T$. If for some $\alpha \subseteq \langle n \rangle$ the matrix $B_0[\alpha]$ is singular, then $B[\alpha]$ is singular.*

Proof. Assume that $B_0[\alpha]$ is singular for some $\alpha \subseteq \langle n \rangle$. Let $y \in N(B_0[\alpha])$. Since B_0 is positive semidefinite, it has the principal submatrix rank property. Hence, by adjoining zero components to y we obtain a vector x in $N(B_0)$. By Corollary (3.12), $x \in N(B)$, and since $\operatorname{supp}(x) \subseteq \alpha$, the matrix $B[\alpha]$ is singular. ∎

PROPOSITION 3.15. *Let A be a Lyapunov diagonally semistable matrix, and let D_0 be a maximal Lyapunov scaling factor of A. Then for every Lyapunov scaling factor D of A the matrix $D_0 + D$ is a maximal Lyapunov scaling factor of A.*

Proof. By part (ii) of Lemma 3.2 we have $V(D_0, A) \subseteq V(D_0 + D, A)$. By the maximality of the dimension of $V(D_0, A)$ we have $V(D_0, A) = V(D_0 + D, A)$. Our assertion now follows from Theorem 3.10. ∎

COROLLARY 3.16. *Let A be a Lyapunov diagonally semistable matrix. Then A has a unique (up to scalar multiplication) maximal Lyapunov scaling factor if and only if A has a unique Lyapunov scaling factor.*

The following lemma is essentially known.

LEMMA 3.17. *Let A be a Lyapunov diagonally semistable matrix. If D is a Lyapunov scaling factor of A, then D^{-1} is a Lyapunov scaling factor of A^T. Furthermore, the dimensions of $V(D, A)$ and $V(D^{-1}, A^T)$ are the same.*

Proof. Let D be a Lyapunov scaling factor of A. The matrix $B = AD + DA^T$ is positive semidefinite, and hence the matrix $D^{-1}BD^{-1} = A^TD^{-1} + D^{-1}A$ is positive semidefinite and has the same rank as B. ∎

COROLLARY 3.18. *Let A be a Lyapunov diagonally semistable matrix. If D is a maximal Lyapunov scaling factor of A, then D^{-1} is a maximal Lyapunov scaling factor of A^T.*

4. LYAPUNOV DIAGONAL SEMISTABILITY OF BLOCK TRIANGULAR MATRICES

In this section we apply the results of the previous section in order to characterize block triangular matrices which are Lyapunov diagonally semistable.

LEMMA 4.1. *Let A be a Lyapunov diagonally semistable matrix, and let D be a Lyapunov scaling factor of A. Then $x \in V(D, A)$ if and only if $D^{-1}x \in V(D^{-1}, A^T)$.*

Proof. Observe that

$$x \in V(D, A) = \mathrm{Range}(AD + DA^T)$$

is equivalent to

$$D^{-1}x \in \mathrm{Range}\big(D^{-1}(AD + DA^T)\big).$$

Since

$$\begin{aligned}\mathrm{Range}\big(D^{-1}(AD + DA^T)\big) &= \mathrm{Range}\big(D^{-1}(AD + DA^T)D^{-1}\big)\\ &= \mathrm{Range}(A^TD^{-1} + D^{-1}A) = V(D^{-1}, A^T),\end{aligned}$$

our claim follows. ∎

Corollary 4.2. *Let A be a Lyapunov diagonally semistable matrix, and let D be a maximal Lyapunov scaling factor of A. Then $x \in V_{A^T}$ if and only if $Dx \in V_A$.*

Proof. The assertion follows immediately from Lemma 4.1 and Corollary 3.18. ∎

In the sequel we assume that the matrix A is a real $n \times n$ matrix given in a $p \times p$ block triangular form $[A_{ij}]$ with square diagonal blocks. The set of indices that correspond to the diagonal block A_{ii} is denoted by α_i, $i \in \langle p \rangle$.

Proposition 4.3. *If the diagonal blocks $A_{11}, \dots, A_{pp}$ are all Lyapunov diagonally semistable matrices, and if*

$$\text{(4.4)} \qquad \mathrm{col}(A_{ij}) \subseteq V_{A_{ii}}, \quad \mathrm{col}\big(A_{ij}^T\big) \subseteq V_{A_{jj}^T}, \qquad i, j \in \langle p \rangle, \quad i \neq j,$$

then there exists a positive diagonal matrix D which is a direct sum of maximal Lyapunov scaling factor of $A_{11}, \dots, A_{pp}$, such that the matrix $B = AD + DA^T$ satisfies

$$\text{(4.5)} \qquad \mathrm{sign}\det B[\beta] = \prod_{i=1}^{p} \mathrm{sign}\det B[\beta \cap \alpha_i]$$

for every $\beta \subseteq \langle n \rangle$.

Proof. Without loss of generality we assume that A is given in an upper block triangular form. We prove our assertion by induction on p. For $p=1$ there is nothing to prove. Assume our assertion holds for $p<m$, where $m>1$, and let $p=m$. Let $\beta \subseteq \langle n \rangle$, and denote $\beta_i = \beta \cap \alpha_i$, $i \in \langle p \rangle$. Let $A^{\sim} = A(\alpha_p)$. By the inductive assumption there exists a positive diagonal matrix $D^{\sim}$ which is a direct sum of maximal Lyapunov scaling factors $D_1, \dots, D_{p-1}$ of $A_{11}, \dots, A_{p-1,p-1}$ respectively, such that the matrix $B^{\sim} = A^{\sim}D^{\sim} + D^{\sim}A^{\sim T}$ satisfies

$$\text{(4.6)} \qquad \operatorname{sign}\det B^{\sim}[\gamma \setminus \alpha_p] = \prod_{i=1}^{p-1} \operatorname{sign}\det B^{\sim}[\gamma \cap \alpha_i]$$

for every $\gamma \subseteq \langle n \rangle$. Let $\hat{D}_p$ be a maximal Lyapunov scaling factor of A_{pp}. Clearly, $D_p = c\hat{D}_p$ is a maximal Lyapunov scaling factor of A_{pp} whenever c is a positive scalar. Let $D_c = D^{\sim} \oplus D_p$, and let $B_c = AD_c + D_cA^T$. Distinguish between two cases:

(1) *The matrix $B_c[\beta_i]$ is singular for some $i \in \langle p \rangle$.* Since $B_c[\alpha_i]$ is positive semidefinite, it has the principal submatrix rank property and hence

$$\text{(4.7)} \qquad \operatorname{Rank}(B_c[\beta_i \mid \alpha_i]) = \operatorname{Rank}(B_c[\beta_i]) < |\beta_i|.$$

Now let $t \in \langle n \rangle$, $t \notin \alpha_i$. By (4.4) we have

$$\text{(4.8)} \qquad A[\alpha_i \mid t] \in V_{A_{ii}} = \operatorname{Range}(B_c[\alpha_i])$$

Also, by (4.4) we have

$$A[t \mid \alpha_i]^T \in V_{A_{ii}^T},$$

and by Corollary 4.2,

$$\text{(4.9)} \qquad D_iA[t \mid \alpha_i]^T \in V_{A_{ii}} = \operatorname{Range}(B_c[\alpha_i]).$$

Observe that

$$B_c[\alpha_i \mid t] = (D_c)_{jj}A[\alpha_i \mid t] + D_iA[t \mid \alpha_i]^T.$$

By (4.8) and (4.9) we now have

$$B_c[\alpha_i | t] \in \mathrm{Range}(B_c[\alpha_i]),$$

and therefore, $B_c[\beta_i \mid t] \in \mathrm{Range}(B_c[\beta_i \mid \alpha_i])$. Thus, $B_c[\beta_i \mid \beta] \in \mathrm{Range}(B_c[\beta_i \mid \alpha_i])$, and by (4.7) the matrix $B_c[\beta]$ is singular.

(2) *The matrices $B_c[\beta_i]$, $i \in \langle p \rangle$, are all nonsingular.* By (4.6), the matrix $B^{\sim}[\beta \setminus \beta_p]$ has a positive determinant. Observe that

$$B_c = \begin{bmatrix} B^{\sim} & cB_{12} \\ cB_{12}^T & cB_{22} \end{bmatrix},$$

where

$$B_{12} = \begin{bmatrix} A_{1p} \hat{D}_p \\ A_{2p} \hat{D}_p \\ \vdots \\ A_{p-1,p} \hat{D}_p \end{bmatrix}$$

and

$$B_{22} = \left[A_{pp} \hat{D}_p + \hat{D}_p A_{pp}^T \right].$$

Thus,

$$\det B_c[\beta] = c^{|\beta_p|} \det \begin{bmatrix} B^{\sim}[\beta \setminus \beta_p] & B_{12}[\beta \setminus \beta_p \mid \beta_p] \\ cB_{12}[\beta \setminus \beta_p \mid \beta_p]^T & B_{22}[\beta_p] \end{bmatrix}.$$

Since $B^{\sim}[\beta \setminus \beta_p]$ and $B_{22}[\beta_p]$ have positive determinants, it follows by continuity arguments that for c sufficiently small the determinant of $B_c[\beta]$ is positive.

We have proven that for a given $B \subseteq \langle n \rangle$, we have

$$\operatorname{sign} \det B_c[\beta] = \prod_{i=1}^{p} \operatorname{sign} \det B_c[\beta \cap \alpha_i]$$

whenever c is sufficiently small. Since the number of subsets β of $\langle n\rangle$ is finite, it follows that for c sufficiently small the matrix $D=D_c$ satisfies the required conditions. ∎

The following theorem follows from Proposition 4.3.

THEOREM 4.10. *Let A be a matrix in a $p\times p$ block triangular form with square diagonal blocks $A_{11},\dots,A_{pp}$. Then A is Lyapunov diagonally semistable if and only if the matrices $A_{11},\dots,A_{pp}$ are Lyapunov diagonally semistable, and*

$$(4.11)\qquad \operatorname{col}(A_{ij})\subseteq V_{A_{ii}},\quad \operatorname{col}(A_{ij}^T)\subseteq V_{A_{jj}^T},\qquad i,j\in\langle p\rangle,\quad i\neq j.$$

Proof. The "if" direction follows from Proposition 4.3. Conversely, suppose that A is Lyapunov diagonally semistable, and let D be a Lyapunov scaling factor of A. Then clearly the matrices $A_{11},\dots,A_{pp}$ are Lyapunov diagonally semistable. Furthermore, we have $D=D_1\oplus\cdots\oplus D_p$, where D_i is a Lyapunov scaling factor of A_{ii}, $i\in\langle p\rangle$. Since $AD+DA^T$ is a positive semidefinite matrix, it has the principal submatrix rank property. Thus, since A is in block triangular form, it follows that

$$(4.12)\qquad \operatorname{col}(A_{ij})\subseteq V(D_i,A_{ii})\subseteq V_{A_{ii}},\qquad i,j\in\langle p\rangle,\quad i\neq j,$$

and

$$\operatorname{col}(D_jA_{ij}^T)\subseteq V(D_j,A_{jj}),\qquad i,j\in\langle p\rangle,\quad i\neq j.$$

By Lemma 4.1 we now have

$$(4.13)\qquad \operatorname{col}(A_{ij}^T)\subseteq V(D_j^{-1},A_{jj}^T)\subseteq V_{A_{jj}^T},\qquad i,j\in\langle p\rangle,\quad i\neq j.$$

Since (4.12) together with (4.13) is (4.11), our theorem is proved. ∎

COROLLARY 4.14. *Let A be a Lyapunov diagonally semistable matrix in a $p\times p$ block triangular form with square diagonal blocks $A_{11},\dots,A_{pp}$. Then there exists a Lyapunov scaling factor D of A which is a direct sum of maximal Lyapunov scaling factors of $A_{11},\dots,A_{pp}$, such that the matrix $B=AD+DA^T$ satisfies (4.5) for every $\beta\subseteq\langle n\rangle$.*

Proof. By Theorem 4.10 the matrices $A_{11},\dots,A_{pp}$ are Lyapunov diagonally semistable, and also A satisfies (4.11). Our assertion now follows from Proposition 4.3. ∎

In the next section we shall show that the matrix D whose existence is asserted in Proposition 4.3 and in Corollary 4.14 is a maximal Lyapunov scaling factor of A. Also we shall show that the equality (4.5) holds for every maximal Lyapunov scaling factor D of A.

An important corollary of Theorem 4.10 is the following.

THEOREM 4.15. *Let A be a matrix in a $p\times p$ (upper) block triangular form with square diagonal blocks $A_{11},\dots,A_{pp}$. Then A is Lyapunov diagonally semistable if and only if for every $i,j\in\langle p\rangle$, $i<j$, the matrix*

$$A^{ij}=\begin{bmatrix} A_{ii} & A_{ij} \\ 0 & A_{jj}\end{bmatrix}$$

is Lyapunov diagonally semistable.

Proof. If A is Lyapunov diagonally semistable, then A^{ij} is Lyapunov diagonally semistable, being a principal submatrix of A. Conversely, if for every $i,j\in\langle p\rangle$, $i<j$, the matrix A^{ij} is Lyapunov diagonally semistable, then the matrices A_{ii} and A_{jj} are Lyapunov diagonally semistable. Furthermore, by applying Theorem 4.10 to all such matrices A^{ij} we get (4.11). It now follows from Theorem 4.10 that the matrix A is Lyapunov diagonally semistable. ∎

It follows from Theorem 4.15 that the problem of characterizing Lyapunov diagonally semistable matrices with p diagonal blocks in the Frobenius normal form can be reduced to the problem of characterizing Lyapunov diagonally semistable matrices with two diagonal blocks in the Frobenius normal form. This observation might be very useful in the investigation of reducible Lyapunov diagonally semistable matrices.

5. FURTHER PROPERTIES OF MAXIMAL LYAPUNOV SCALING FACTORS AND LYAPUNOV RANGES

In this section we discuss several corollaries to our main results, introduced in the previous section. We use Lemma (4.1) of [4], which is quoted here in a slightly different formulation.

LEMMA 5.1. *Let an $n \times n$ matrix A have the principal submatrix rank property, and let $\mu_1, \dots, \mu_q$ be pairwise disjoint nonempty sets such that*

$$\bigcup_{i=1}^{q} \mu_i = \langle n \rangle.$$

Then

$$\operatorname{Rank}(A) \le \sum_{i=1}^{q} \operatorname{Rank}(A[\mu_i]). \tag{5.2}$$

Furthermore, equality holds in (5.2) if and only if for every nonempty subsets $\gamma_1, \dots, \gamma_q$ of $\mu_1, \dots, \mu_q$ respectively, we have

$$\operatorname{Rank}\left(A\left[\bigcup_{i=1}^{q} \gamma_i\right]\right) = \sum_{i=1}^{q} \operatorname{Rank}(A[\gamma_i]).$$

PROPOSITION 5.3. *Let A be a Lyapunov diagonally semistable matrix in a $p \times p$ block triangular form with square diagonal blocks $A_{11}, \dots, A_{pp}$, and let D be a Lyapunov scaling factor of A. Then*

$$\dim(V(D, A)) \le \sum_{i=1}^{p} \dim(V(D[\alpha_i], A_{ii})).$$

Proof. Since $B = AD + DA^T$ is a positive semidefinite matrix, it has the principal submatrix rank property. Our claim now follows from Lemma 5.1. ∎

PROPOSITION 5.4. *Let A be a Lyapunov diagonally semistable matrix in a $p \times p$ block triangular form with square diagonal blocks $A_{11}, \dots, A_{pp}$. Let D be a Lyapunov scaling factor of A, and let $B = AD + DA^T$. Then*

$$\dim(V(D, A)) = \sum_{i=1}^{p} \dim(V(D[\alpha_i], A_{ii})) \tag{5.5}$$

if and only if (4.5) holds for every $B \subseteq \langle n \rangle$.

Proof. We first prove the "if" part of our proposition. Suppose that (4.5) holds for every $\beta \subseteq \langle n \rangle$. In view of Proposition 5.3 it is enough to prove that

$$\dim(V(D, A)) \geq \sum_{i=1}^{p} \dim(V(D[\alpha_i], A_{ii})).$$

Let $\beta_i \subseteq \alpha_i$ be such that $B[\beta_i]$ is a largest nonsingular submatrix of $B[\alpha_i]$. Observe that $|\beta_i| = \dim(V(D[\alpha_i], A_{ii}))$. Let β be the subset of $\langle n \rangle$ such that $\beta \cap \alpha_i = \beta_i$, $i \in \langle n \rangle$. By (4.5), $B[\beta]$ is nonsingular, and thus indeed

$$\dim(V(D, A)) = \operatorname{Rank}(B) \geq |\beta| = \sum_{i=1}^{p} |\beta \cap \alpha_i| = \sum_{i=1}^{p} \dim(V(D[\alpha_i], A_{ii})).$$

To prove the "only if" part let (5.5) hold. Let $\beta \subseteq \langle n \rangle$. Since the matrix B is positive semidefinite, it follows that if $B[\beta]$ is nonsingular then $B[\beta \cap \alpha_i]$ is nonsingular for every $i \in \langle p \rangle$. Therefore, in order to prove (4.5), all we have to show is that if $B[\beta \cap \alpha_i]$ is nonsingular for every $i \in \langle p \rangle$, then $B[\beta]$ is nonsingular. This follows from (5.5) by Lemma 5.1. ■

REMARK 5.6. We remark that the condition that A is in block triangular form can be removed from Propositions 5.3 and 5.4. These propositions are stated in the present form to fit with the rest of the paper.

COROLLARY 5.7. *Let A be a Lyapunov diagonally semistable matrix in a $p \times p$ block triangular form with square diagonal blocks $A_{11}, \dots, A_{pp}$. Then*

$$\dim(V_A) = \sum_{i=1}^{p} \dim(V_{A_{ii}}).$$

Proof. Proposition 5.3 yields that

$$\dim(V_A) \leq \sum_{i=1}^{p} \dim(V_{A_{ii}}). \tag{5.8}$$

By Corollary 4.14 and Proposition 5.4 there exists a Lyapunov scaling factor D of A such that

$$\dim(V(D, A)) = \sum_{i=1}^{p} \dim(V_{A_{ii}}). \tag{5.9}$$

Since $\dim(V_A) \geq \dim(V(D, A))$, our result follows from (5.8) and (5.9). ■

The following well-known result (e.g. [1]) now becomes an immediate consequence of Corollary 5.7.

COROLLARY 5.10. *Let A be a matrix in a $p \times p$ block triangular form with square diagonal blocks $A_{11}, \dots, A_{pp}$. Then A is Lyapunov diagonally stable if and only if the matrices $A_{11}, \dots, A_{pp}$ are Lyapunov diagonally stable.*

Another interesting consequence of the above results is the following characterization of maximal Lyapunov scaling factors.

THEOREM 5.11. *Let A be a Lyapunov diagonally semistable matrix in a $p \times p$ block triangular form with square diagonal blocks $A_{11}, \dots, A_{pp}$, and let D be a Lyapunov scaling factor of A. Then D is a maximal Lyapunov scaling factor of A if and only if $D[\alpha_i]$ is a maximal Lyapunov scaling factor of A_{ii}, $i \in \langle p \rangle$, and the matrix $B = AD + DA^T$ satisfies (4.5) for every $\beta \subseteq \langle n \rangle$.*

Proof. Suppose that D is a maximal Lyapunov scaling factor of A. By Proposition 5.3 and Corollary 5.7 we have

$$\text{(5.12)} \quad \sum_{i=1}^{p} \dim(V_{A_{ii}}) = \dim(V(D, A)) \leq \sum_{i=1}^{p} \dim(V(D[\alpha_i], A_{ii})).$$

Since we have $\dim(V(D[\alpha_i], A_{ii})) \leq \dim(V_{A_{ii}})$ for every $i \in \langle p \rangle$, it follows that (5.12) holds if and only if for every $i \in \langle p \rangle$ we have $\dim(V(D[\alpha_i], A_{ii})) = \dim(V_{A_{ii}})$, which means that $D[\alpha_i]$ is a maximal Lyapunov scaling factor of A_{ii}. By Proposition 5.4, the equality (4.5) is satisfied for every $\beta \subseteq \langle n \rangle$.

Conversely, suppose that $D[\alpha_i]$ is a maximal Lyapunov scaling factor of A_{ii}, $i \in \langle p \rangle$, and that the matrix $B = AD + DA^T$ satisfies (4.5) for every $\beta \subseteq \langle n \rangle$. By Proposition 5.4 we have

$$\dim(V(D, A)) = \sum_{i=1}^{p} \dim(V_{A_{ii}}),$$

and by Corollary 5.7, D is a maximal Lyapunov scaling factor of A. ■

Observe that in view of Theorem 5.11, the matrix D whose existence is asserted in Proposition 4.3 and in Corollary 4.14 is a maximal Lyapunov scaling factor of A.

We remark that the condition that $D[\alpha_i]$ is a maximal Lyapunov scaling factor of A_{ii}, $i \in \langle p \rangle$, is not sufficient by itself to have D be a maximal Lyapunov scaling factor of A. This is demonstrated by the following example.

EXAMPLE 5.13. Let A be the upper triangular matrix

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Observe that the identity matrix of order 1 is a maximal Lyapunov scaling factor of both A_{11} and A_{22}. Nevertheless, the identity matrix of order 2 is not a maximal Lyapunov scaling factor of A.

We also remark that Theorem 5.11 does not hold in general for matrices not in block triangular form. Namely, if D is a maximal Lyapunov scaling factor of an $n \times n$ matrix A and if α is a proper subset of $\langle n \rangle$, then $D[\alpha]$ need not be a maximal Lyapunov scaling factor of $A[\alpha]$, as demonstrated by the following example.

EXAMPLE 5.14. Let A be matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}.$$

As observed in [2], the identity matrix is a unique, and thus a maximal, Lyapunov scaling factor of A. However, since $C = A[\{1,2\}]$ is Lyapunov diagonally stable, and since $C + C^T$ is singular, it follows that the 2×2 identity matrix is not a maximal Lyapunov scaling factor of C.

We continue with a lemma that is essentially observed in the proof of Theorem 4.35 in [3].

LEMMA 5.15. *Let A be a Lyapunov diagonally semistable matrix, and let D be a Lyapunov scaling factor of A. Then $V(D, A) \subseteq \mathrm{Range}(A)$.*

Proof. By Lemma 6.6 in [2] we have

$$N(A^T) \subseteq N(AD + DA^T). \tag{5.16}$$

By statements (ii) and (iii) of Proposition (3.1), our claim follows from (5.16). ∎

Corollary 5.17. *Let A be a Lyapunov diagonally semistable matrix. Then* $V_A \subseteq \text{Range}(A)$.

Note that, by Corollary (5.17), the condition (4.11) is related to but somewhat stronger than the condition that is yielded by the weak principal submatrix rank property. The latter is proved in Theorem 4.35 of [3] to be shared by Lyapunov diagonally semistable matrices, and it yields that

(5.18)

$$\text{Col}(A_{ij}) \subseteq \text{Range}(A_{ii}), \quad \text{Col}(A^T_{ij}) \subseteq \text{Range}(A^T_{jj}), \qquad i, j \in \langle p \rangle, \quad i \neq j.$$

However, if $\dim(V_{A_{ii}}) = \text{Rank}(A_{ii})$, $i \in \langle p \rangle$, then (5.18) and (4.11) are identical, and hence we have

Corollary 5.19. *Let A be a real matrix in a $p \times p$ block triangular form with square diagonal blocks $A_{11}, \dots, A_{pp}$. Suppose that the diagonal blocks are all Lyapunov diagonally semistable and that* $\dim(V_{A_{ii}}) = \text{Rank}(A_{ii})$, $i \in \langle p \rangle$. *Then A is Lyapunov diagonally semistable if and only if A has the weak principal submatrix rank property.*

This observation raises the natural question of when we have the equality

$$V_A = \text{Range}(A). \tag{5.20}$$

Examples of such cases are the following:

(1) Let A be an irreducible $n \times n$ M-matrix. Then, as is well known, $\text{Rank}(A) \geq n - 1$. Observe that for a Lyapunov scaling factor D of A, the matrix $AD + DA^T$ is also an irreducible M-matrix. Hence, it follows that the $\dim(V_A) \geq n - 1$. Furthermore, if A is nonsingular, then A is Lyapunov diagonally stable and hence V_A is n-dimensional. Thus, it follows that (5.20) holds.

(2) Let A be an irreducible $n \times n$ H_+-matrix. If A is nonsingular, then by Theorem 3.19 in [3], A is Lyapunov diagonally stable. If A is singular, then by Proposition 3.1 in [3], A is Lyapunov diagonally semistable. Further, by Lemma 3.20 in [3], all proper principal submatrices of A are nonsingular. Hence, by Theorem 4.14(v) in [3], $\dim(V_A) = \text{Rank}(A) = n - 1$, and so (5.20) holds. In view of this observation, Corollary (5.19), when applied to H_+-matrices, yields some of the implications of Theorem 4.14 in [3].

(3) Let A be a rank one Lyapunov diagonally semistable $n \times n$ matrix. We first prove that A must have a nonzero diagonal element. Since A is of rank one, there exist $i, j \in \langle n \rangle$ such that $a_{ij} \neq 0$. If $a_{ii} = a_{jj} = 0$ then, since $\text{Rank}(A) = 1$, we have $a_{ji} = 0$. But then the principal minor of $AD + DA^T$ based on $\{i, j\}$ is negative for every nonsingular real diagonal matrix D, in contradiction to the Lyapunov diagonal semistability of A. Since A has a nonzero diagonal element, it now follows that $\dim(V_A) = \text{Rank}(A) = 1$, and thus we have (5.20).

(4) Let A be a positive semidefinite (symmetric) matrix. Then I is a Lyapunov scaling factor of A. Since $V(I, A) = \text{Range}(A)$, it follows from Corollary 5.17 that (5.20) holds.

However, (5.20) does not hold in general, as demonstrated by the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

It is easy to verify that $V_A = \{0\}$, while $\text{Range}(A) = R^2$.

REFERENCES

1 A. Berman and D. Hershkowitz, Matrix diagonal stability and its implications, *SIAM J. Algebraic Discrete Methods* 4:377–382 (1983).

2 D. Hershkowitz and H. Schneider, Scalings of vector spaces and the uniqueness of Lyapunov scaling factors, *Linear and Multilinear Algebra* 17:203–226 (1985).

3 D. Hershkowitz and H. Schneider, Lyapunov diagonal semistability of real *H*-matrices, *Linear Algebra Appl.* 71:119–149 (1985).

4 D. Hershkowitz and H. Schneider, On Lyapunov scaling factors for real symmetric matrices, *Linear and Multilinear Algebra* to appear.

Received 26 February 1987; revised 17 June 1987